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Classification of 4-point regular triangle-free graphs

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1 Introduction

In this paper, we use planar algebra techniques to classify a certain type of highly symmetric graph. These types of graphs played a key role in completing the classification of finite simple groups in [Gor86]. More recently, these graphs have been used to classify spin models, which originated from physics but are used now in the field of quantum algebra to discover new Hopf algebras.

A graph is n-point regular if the number of vertices connected to any n vertices is only dependent on how they connect to each other (See Section 2 for the full definition). This is a weaker relation than n-transitivity. 1-point and 2-point regular graphs—called regular and strongly regular graphs, respectively—have been well studied. Aside from a disjoint union of complete graphs (and their complements) the only 5-point regular or higher graph is the pentagon.

Little headway has been made into the classification of 3-point regular and 4-point regular graphs. However, in [CGS78], a partial categorization of 3-point regular graphs was given. Aside from a disjoint union of complete graphs (and their complements), there are no known examples of 3-point or 4-point regular graphs of 100 vertices or more. In this paper, we classify all 4-point regular triangle-free graphs.

As mentioned above, these highly symmetric graphs also appear in seemingly disparate places. In [Jae95], Jaeger noticed that spin models for the Kauffman polynomial—a well-known knot polynomial—were connected to certain 3-point regular graphs. In [Edg19], Edge expanded this classification to show that every 3-point regular graph meeting a certain condition gives a spin model for a singly-generated Yang-Baxter planar algebra, which can be thought of as a slight abstraction of a knot polynomial.

In this paper we utilize the framework created in [Edg19] to classify all 4-point regular triangle-free graphs:

Theorem 1. *Let Γ be a 4-point regular triangle free graph. Then it is one of the following:*

- *A Complete Bipartite Graph*
- *A Collection of Bars*
- *A Collection of Vertices Without Edges*
- *The pentagon*

The benefit to using a planar algebra framework is that it provides a quick and tactile proof of the classification of this class of graph. It is our hope that we will expand this classification to include other special types of 4-point regular graphs (e.g. claw-free, Λ-free graphs) in the future.

2 Graph Theory

The following definitions are standard from graph theory:

Definition 2.1. A graph $\Gamma = (V, E)$ where V is a vertex set and E a set containing pairs of vertices which we call edges. We also write $V(\Gamma)$ to represent the vertex set of Γ, and $E(\Gamma)$ to represent the edge set of Γ. Furthermore if two vertices $a, b \in V$ share an edge between so $(a, b) \in E$, then we say that a and b are adjacent, or equivalently a and b are neighbors.

In this paper, we will only consider graphs in the traditional sense of the term; two vertices can only have at most one edge between them and no self-loops are allowed. If there is a vertex x that is adjacent to a vertex a and and a vertex b, we say that a and b have a *common neighbor* x.

Definition 2.2. The adjacency matrix of a graph Γ is a matrix, M, with rows and columns indexed by the vertices. If vertex i and j share an edge, then $M_{i,j} = 1$. If two vertices do not share an edge $M_{i,j} = 0$.

Because we only consider graphs in the sense mentioned above, all the adjacency matrices will only contain 0s and 1s, with 0s along the diagonal.

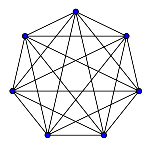
Definition 2.3. Let Γ be a graph. The graph complement of Γ, Γ^c, is a graph composed of the same number of vertices as Γ with the property that for all $i \neq j$, $(v_i, v_j) \in E(\Gamma^c)$ if and only if $(v_i, v_j) \notin E(\Gamma)$.

Definition 2.4. Let Γ be a graph. The full subgraph of Γ generated by vertices $A = \{v_1, \ldots, v_n\} \subseteq V(\Gamma)$ is a graph Γ_A where $V(\Gamma_A) = A$ and $E(\Gamma_A) = \{(v_i, v_j) | (v_i, v_j) \in E(\Gamma) \text{ for } v_i, v_j \in A\}$.

Definition 2.5. A graph Γ is said to be triangle-free if the number of edges between any three vertices in Γ is less than or equal to two.

Definition 2.6. A complete graph K_n is a graph with n vertices, where every vertex is adjacent to every other vertex.

Below is the graph of K_7:



Definition 2.7. A graph is said to be bipartite if it can be broken up into two different sets of vertices V_i and V_j, where none of the vertices in V_i are adjacent, and none of the vertices in V_j are adjacent. Note, we will also refer to these vertex sets V_i and V_j as V_+ and V_- respectively. A complete bipartite graph $K_{i,j}$ is a bipartite graph where every vertex in V_i is adjacent to every vertex in V_j.

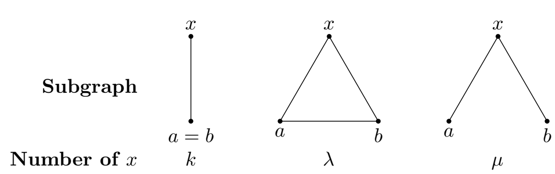
The following definition is from [Kup97]:

Definition 2.8. A graph Γ is n-point regular if the number of common neighbors of n vertices in Γ depends only on how the vertices connect. That is, if the full subgraph generated by two sets of n vertices are isomorphic, then the number of common neighbors of the first set is equal to the number of common neighbors of the second set.

This special class of highly symmetric graph has many other names including $C(n)$ and graphs satisfying the $n + 1$ vertex condition.

Example 2.9. A graph is 1-point regular if the number of common neighbors for any vertex is the same for all vertices. 1-point regularity is often just called regularity, and typically we use the parameter k to represent the number of common neighbors.

Example 2.10. A graph is 2-point regular—also called strongly regular—if the number of common neighbors of any 2 vertices only depends on how the vertices connect. For any two distinct vertices a and b, the number of possible ways that they can relate is shown below



where the parameters for the number of common neighbors are k, λ and μ. A strongly regular graph with n vertices is often denoted by its parameters, $\text{srg}(n, k, \lambda, \mu)$.

Notice here that a 2-point regular graph is necessarily 1-point regular by definition. Let $m < n$. Then, it is clear that an n-point regular graph is also m-point regular by definition if one only considers m of the n points to be distinct.

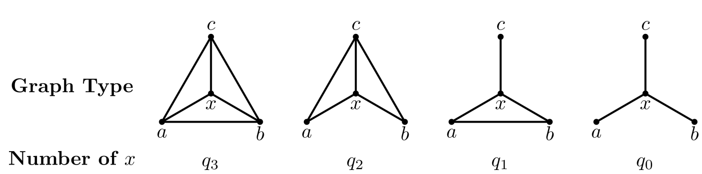
Lemma 2.11. *If $K_{i,j}$ is a strongly regular graph then $i = j$. Moreover, a strongly regular triangle-free graph is isomorphic to $K_{i,i}$ if and only if $n = 2k$, where n is the number of vertices in $K_{i,j}$, and k is the parameter k show in Example 2.10.*

Proof. Let $K_{i,j}$ be a bipartite graph that is strongly regular. Thus, $K_{i,j}$ is also regular, so each vertex has the same number of neighbors. In particular, if x and y are vertices that are adjacent, they also have the same number of neighbors. Without loss of generality, we can assume that x has i neighbors and y has j neighbors, by definition of $K_{i,j}$. Thus, $i = j$, as desired.

Notice that $K_{i,i}$ is strongly regular, and by inspection $n = 2k$. Let Γ be a strongly regular, triangle-free graph with $n = 2k$. Let a be a vertex in Γ. Because $n = 2k$, then a is adjacent to half of the vertices in Γ. Now, let b and c be adjacent vertices in Γ such that they are not adjacent to a. Notice that b and c have no common neighbors, else they would form a triangle. Furthermore, b must be adjacent to k vertices. Excluding a there are $k - 1$ vertices in Γ which b is not adjacent to. Similarly, c must also be adjacent to k vertices. Then because b and c share no common neighbors c must be adjacent to a, which is a contradiction. Thus if b and c are not adjacent to a, it follows that b and c are not adjacent. Thus, any vertex not adjacent to a must be adjacent to the same k vertices as a, and any vertex adjacent to a must be adjacent to the k vertices a is not adjacent to. Therefore Γ is $K_{i,i}$ as desired. $\square$

1-point and 2-point regular graphs have been studied extensively. Less is known, however, about 3-point regular graphs and 4-point regular graphs (See [CGS78] for a categorization of 3-point regular graphs.)

Example 2.12. A graph is 3-point regular if it is 2-point regular and for any three distinct vertices a, b, c the number of possible ways that they can relate is shown below:

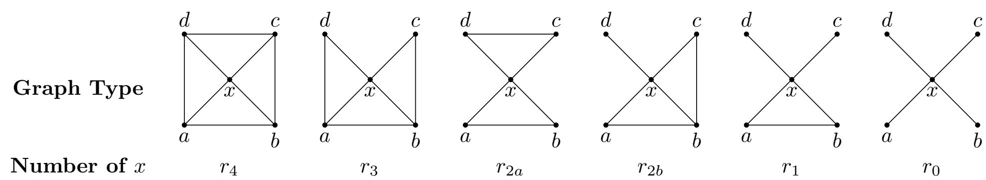


The following theorem is well-documented:

Theorem 2.13. *Let Γ be a n-point regular graph where $n \geq 5$ such that both Γ and its complement are connected. Then Γ is isomorphic to the pentagon.*

Thus, progress has been made on n-point regular graphs for all $n \neq 4$. In this paper, we will shed light on this class of graphs.

Example 2.14. Γ is 4-point regular if it is 3-point regular and for any four distinct vertices a, b, c and d in Γ, the number of possible subgraphs for vertices a, b, c, and d depend only on how those four points relate. The different possible ways they can relate are shown below:



Lemma 2.15. *Let Γ be a 4-point regular, triangle-free graph and let a, b, c, d be vertices in Γ that are all adjacent to the vertex x. Then a, b, c, d are pairwise non-adjacent.*

Proof. Let Γ be a 4-point regular, triangle-free graph and $a, b, c, d \in V(\Gamma)$ and x be a vertex connected to a, b, c and d. Because Γ is triangle-free, a, b, c and d must be pairwise non-adjacent, else the two points and x would form a triangle. □

Lemma 2.16. *Let Γ be a 4-point regular triangle-free graph where $r_0 = 0$. Then $k < 4$, where k is the parameter shown in Example 2.10.*

Proof. Let Γ be a 4-point regular triangle-free graph with $r_0 = 0$. Let a be a vertex in Γ and choose $k \geq 4$. Thus a must be adjacent to four vertices. Because Γ is triangle-free then r_4, r_3, r_2, and r_1 are all equal to zero. But this implies that $r_0 \neq 0$, which is a contradiction. Thus, it follows that $k < 4$. □

Proposition 2.17. *Let Γ be a 4-point regular, triangle-free graph with parameters shown in Examples 2.9-2.12. Then $k \geq \mu \geq q_0 \geq r_0 \geq 0$ and all other parameters are 0.*

Proof. Let Γ be a triangle-free graph. By definition, $k, \mu, q_0, r_0 \geq 0$. For all other parameters, note that if any of them were non-zero a triangle would necessarily exist in the graph, a contradiction. Thus, they must all be 0. Let a and b be distinct vertices in Γ, let T be the set of common neighbors of a and b, and let S be the set of vertices adjacent to a. By definition $|S| = \mu$, $|T| = k$, and $S \subset T$. Thus $|T| \geq |S|$, so $k \geq \mu$.

Let a, b, c, d be four vertices in Γ, let T be the set of common neighbors of a and b, and let S be the set of common vertices adjacent to a, b, and c. For all $s \in S$, it must be the case that $s \in T$. Since $|T| = \mu$ must be greater than or equal to $|S| = q_0$, then $\mu \geq q_0$. Now, let K be the set of common neighbors of a, b, c, and d. For all $k \in K$, it must be the case that $k \in S$. Since $|S| = q_0$ must be greater than or equal to $|K| = r_0$, then $q_0 \geq r_0$.

Thus, $k \geq \mu \geq q_0 \geq r_0$, as desired. □

Proposition 2.18. *Let Γ be a graph in the following list:*

1. *A collection of vertices without edges,*
2. *A collection of bars, a disjoint union of a finite number of K_2 subgraphs,*
3. *the pentagon, or*
4. *$K_{i,i}$.*

Then Γ is 4-point regular and triangle-free.

Proof. By Theorem 2.13, we know a collection of vertices with no edges and the pentagon are 5-point regular, so they are also 4-point regular. Let Γ be a collection of bars. Let a and b be adjacent vertices in Γ. Thus, a and b share no common neighbors. Therefore $\lambda = 0$ and Γ is triangle-free as desired. Because each vertex in Γ is only adjacent to one other vertex in Γ we have that $k = 1$ and all other parameters are 0. Therefore Γ is a 4-point regular triangle-free graph.

Let $\Gamma = K_{i,i}$ and consider the case where $i \leq 3$. $K_{1,1}$ is a collection of bars and thus 4-point regular and triangle-free by above. $K_{2,2}$ is triangle-free and has parameters $k = 2$ and $\mu = 2$, with all other parameters 0. $K_{3,3}$ is triangle-free and has parameters $k = 3$ and $\mu = 3$, with all other parameters 0. Thus, in each case Γ is 4-point regular and triangle-free.

Let $\Gamma = K_{i,i}$ where $i \geq 4$. Now let x_1 be a vertex in V_+, and let y_1 be a vertex in V_-. Thus, x_1 is necessarily adjacent to y_1. Then because $K_{i,i}$ is complete bipartite for any vertex x_2 in V_+, x_1 is not adjacent to x_2. Similarly for any vertex y_2 in V_-, y_1 is not adjacent to y_2. Consequently x_1 and y_1 share no common neighbors. Therefore $\lambda = 0$ and $K_{i,i}$ is triangle-free. So we need only determine k, μ, r_0, q_0

Now Let a, b, c, d be distinct vertices in, without loss of generality, V_+. By the definition of a complete bipartite graph a is adjacent to every vertex in V_- and so $k = i$. Notice by the definition of a complete bipartite graph a, b, c and d are non-adjacent to one another, but that each vertex is adjacent to every vertex in V_-. Thus, a and b are not adjacent and share i common neighbors, so $\mu = i$. a, b and c are all not adjacent and share i common neighbors, so $q_0 = i$. Finally a, b, c and d are all not adjacent and share i common neighbors, so $r_0 = i$. Therefore, $k = i$,$\mu = i$, $q_0 = i$ and $r_0 = i$, with all other parameters being 0. Thus, in each case Γ is a 4-point regular, triangle-free graph. This completes our proof. □

Lemma 2.19. *Let Γ be a strongly regular triangle-free graph with parameter $\mu = 0$. Then Γ has parameter $k = 0$ or $k = 1$, and Γ is a 4-point regular graph.*

Proof. Let Γ be a strongly regular, triangle-free graph with $\mu = 0$. Suppose that $k \geq 2$, and let a, b, and c, be vertices of Γ such that a is adjacent to both b and c. Since Γ is triangle-free, b and c cannot be adjacent. Thus $\mu > 0$, as b and c share a in common. Thus, we have achieved a contradiction. Therefore, $k = 0$ or $k = 1$. If $k = 0$, then the graph has no edges and if $k = 1$, the graph is a collection of bars, which are both 4-point regular by Proposition 2.18. Thus, any triangle-free, strongly regular graph with $\mu = 0$ is 4-point regular, as desired. □

Lemma 2.20. *Let Γ be a 4-point regular triangle-free graph with $k = 0$ or 1. Then Γ is a graph with no edges or a collection of bars.*

Proof. Let Γ be a 4-point regular, triangle-free graph with $k = 0$ or 1. If $k = 0$, then every vertex has no neighbors, meaning the graph has no edges. If $k = 1$, then each vertex is connected to exactly one other, making the graph a collection of bars, as desired. □

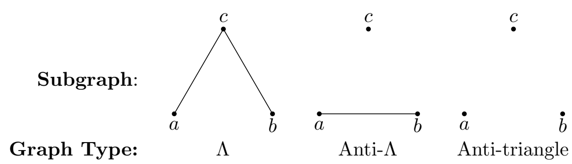
Lemma 2.21. *Let Γ be a 4-point regular triangle-free graph with $k = 2$. Then Γ is the square or the pentagon.*

Proof. Let Γ be a 4-point regular, triangle-free graph with $k = 2$ and let n be the number of vertices of Γ. Necessarily, $4 \leq n \leq 5$. Since Γ is 4-point regular, it is also strongly regular. The list of such graphs is small: K_4, K_5, the square, the pentagon, and their complements. By inspection, the only graphs with $4 \leq n \leq 5$, that are triangle-free and have that $k = 2$, are the square and the pentagon.

Suppose $n \geq 6$. Label the vertices of Γ from x_1 to x_n. And suppose that x_2 is adjacent to x_1 and x_3. Since Γ is triangle-free, without loss of generality, we can assume x_3 is connected to x_4. Continuing this process, we see that necessarily x_n is connected to x_1. Thus, Γ is the n-gon. Notice that x_1 is not adjacent to either x_3 or x_4, since $n \geq 6$. x_1 and x_3 share a vertex, but x_1 and x_4 do not, implying Γ is not 4-point regular, a contradiction. Thus, there are no 4-point regular, triangle-free graphs with $k = 2$, where $n \geq 6$. Therefore, the only two such graphs are the square and the pentagon, as desired. □

Lemma 2.22. *A strongly regular graph with $k = 3$ is a 4-point regular triangle-free graph only when $\Gamma = K_{3,3}$.*

Proof. Let Γ be a strongly regular graph with $k = 3$. Based on Proposition 2.17 and Lemma 2.19, for Γ to be a 4-point regular triangle free graph, the parameter μ is either $1, 2$, or 3. Now suppose there are n vertices in graph Γ. Then there are $\frac{3n}{2}$ edges in Γ. Thus, n has to be an even integer. Then there are $\binom{n}{2} - \frac{3n}{2}$ pairs of nonadjacent vertices. Since each vertex has 3 neighbors, there are $n \cdot \binom{3}{2}$ subgraphs of the following form:



Because Γ is 4-point regular it is also strongly regular. Therefore, each pair of non-adjacent points has the same number of common neighbors. Hence, we have that

$$\mu = \frac{\binom{3}{2} \cdot n}{\binom{n}{2} - \frac{3n}{2}} = \frac{6}{n-4}.$$

Now, we have the following cases: $\mu = 1$ and $n = 10$, $\mu = 2$ and $n = 7$, and $\mu = 3$ and $n = 6$. Since n is determined to be an even integer, only the $\mu = 1$, $n = 10$ and the $\mu = 3$, $n = 6$ cases remain. The $\mu = 1$ case implies that the graph Γ is isomorphic to the Petersen Graph, which is not 3-point regular. Therefore, it is not 4-point regular. The $\mu = 3$ case implies that the graph Γ is isomorphic to the $K_{3,3}$ graph. Lemma 2.18 has shown that $K_{3,3}$ is a 4-point regular, triangle-free graph. Thus, the only such graph is $K_{3,3}$, as desired. ☐

Lemma 2.23. *For a 3-point regular, triangle-free graph Γ where $k \geq 3$, then $q_3 - 3q_2 + 3q_1 - q_0 \neq 0$.*

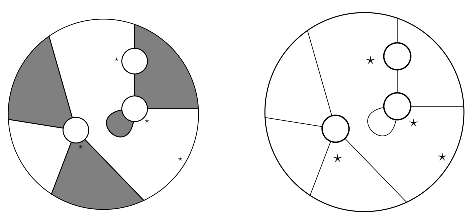
Proof. Let Γ be a 3-point regular triangle-free graph, where $k \geq 3$. Because the graphs q_3, q_2, q_1 all contain a triangle and since Γ is triangle-free, it follows that $q_3 = q_2 = q_1 = 0$. Since $k \geq 3$, there exists a vertex x such that x is adjacent to at three distinct vertices a, b, and c. Therefore $q_0 > 0$, as Γ is 3-point regular. Hence, $q_3 - 3q_2 + 3q_1 - q_0 = -q_0 \neq 0$, as desired. ☐

3 Planar Algebras

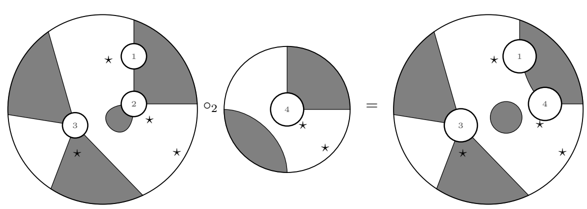
The following definitions are in the styles of [Jon99], [Pet09], and [Edg19]: Note for the purpose of this paper we will define planar algebras as shaded planar algebras, but there does exist a notion of unshaded planar algebras, which is explored more in [Edg19].

To define a planar algebra is it easier to first define and understand the concept of a planar diagram, as well as the role of the planar operad.

Definition 3.1. A planar diagram is a diagram consisting of a unit disc D, often called the output disc, which has a finite number of boundary points. There are also a finite number of input discs, each with a finite number of boundary points. The input discs and the output disc are connected by non-crossing strings. The areas which are divided up by the strings are called the regions of the planar diagram. To indicate the first region we star the region that precedes it, near the boundary component. A planar diagram can be either shaded or unshaded. Unshaded means that the regions are all unshaded. A shaded planar diagram is shaded in a checkerboard fashion. An example of a shaded planar diagram and its unshaded version can be seen below,



Given two planar diagrams A and B, we can compose A with B, if the number of strings on the output disc of B is the same as the number of strings in one of A's input discs. We notate this composition by $A \circ B$. An example of this composition can be seen below,



In this example, the planar diagram with inner disc 4 is inserted into inner disc 2 of the other planar diagram, giving the new planar diagram on the right.

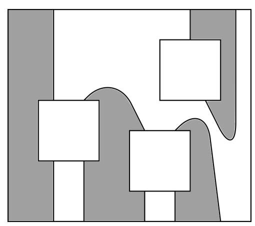
Definition 3.2. The planar operad $\mathcal{P}$ is the set of isotopy classes of planar diagrams under composition.

Definition 3.3. The action of the planar operad is an assignment of a multi-linear vector space map to each planar diagram that is consistent with composition.

We need a way of rewriting the diagrams in a standard form, so that operations can be preformed on these diagrams in a general way. We define this standard form as follows:

Definition 3.4. A planar diagram T is said to be in standard form, when the input and output discs are draw as rectangles, with strings attached only to the top or bottom and the starred region represented by the region bordering the left-hand side of the rectangle. We can draw any planar diagram in standard form up to isomorphism.

Here is an example of a standard diagram,



Definition 3.5. A planar algebra is a family of K vector spaces $\mathcal{V} = V_{\pm i}, i \in \mathbb{Z}_{\geq 0}$, (generally over the complex numbers), together with an action of the planar operad. The action of the planar operad assigns a multilinear map to every $T \in P$. If T is a planar diagram that has k input discs, then the multilinear map would be as follows.

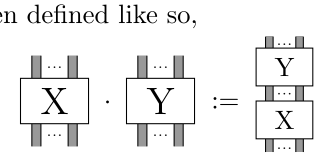
$$T : V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_k} \to V_0$$

so that i_m is the number of strands connected to the boundary of the m-th inner boundary disc, and o is the number of strands connected to the outer boundary disc.

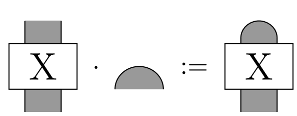
The vector spaces, $V_{\pm n}$ are typically called the n-box spaces. Generally, it is not necessary to distinguish between the two shadings. If this is necessary, we will denote them by V_{+i} and V_{-i}.

We are able to define operations on planar algebras, such as multiplication, capping, rotation, and an inner product. These operations are often defined like so,

Multiplication of two diagrams:



Capping off a diagram on the top:

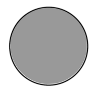


In order to define a inner product we must first define a evaluable planar algebra.

Definition 3.6. A planar algebra $\mathcal{V}$ is called evaluable if dim $V_0 = 1$, or, equivalently, any element of V_0 is equivalent to the empty diagram up to a scalar. Thus, we can define a map from $V_0 \to k$ by sending the empty diagram to 1. We can then use this map to define an inner product on each V_n.

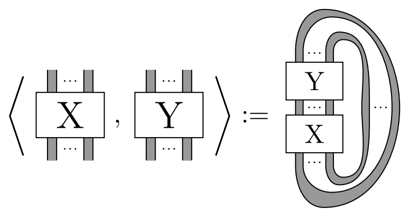
If $\dim V_0 = 1$, then by definition it has the relation stated in the following definition:

Definition 3.7. Let $\mathcal{V}$ be a planar algebra. We say that $\mathcal{V}$ has modulus d if this planar algebra has a relation of the form

 $= d$

where the right side of the equation represents d times the empty diagram.

Definition 3.8. Given an evaluable planar algebra $\mathcal{V}$, the inner product on $\mathcal{V}$ can be defined as,



Note that we could have also connected the strands on the left. If these quantities are the same,the planar algebra is said to be *spherical*.

Definition 3.9. An element $v_n \in V_n$ is said to be a negligible element if $\langle v_n, w_n \rangle = 0$, for all $w_n \in V_n$.

Definition 3.10. A planar algebra is non-degenerate if it has no negligible elements.

In order to find negligible elements V_n, we need a basis for V_n, $\{v_1, v_2, \ldots, v_k\}$. We can then take the matrix of inner products, M, where the (i, j)-th entry of M is the inner product of v_i with v_j. The null space of this matrix would then give us the linear subspace of V_n spanned by the negligible elements. The matrix M is also know as the degeneracy matrix and is vital piece of mathematical machinery for this paper. We want to be be able to transition between graphs to planar diagrams. We do this by creating and isomorphism between the two, and the following definitions are in motivation of this.

Definition 3.11. Let $\mathcal{V}$ and $\mathcal{W}$ be planar algebras. A planar algebra map $\Phi : \mathcal{V} \to \mathcal{W}$ is a collection of linear maps $\phi_{\pm i} : V_{\pm i} \to W_{\pm i}$, intertwining the actions of the planar operad.

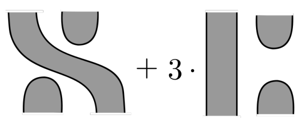
Intertwining the actions of the planar operad, means that $T(v_1, \ldots, v_k) = v_o$ implies
$T(\Phi(v_1), \ldots, \Phi(v_k)) = \Phi(v_o)$ for all $T \in P$ and $v_i \in V$.

4 Planar Algebra Examples

In this section, we will focus on a few common examples of shaded planar algebras. For most of these planar algebras, there are similar unshaded versions. We note here that this paper only focuses on the shaded variety, so we will mean "shaded planar algebra" whenever we say "planar algebra".

Example 4.1. The shaded Temperley-Lieb planar algebra is the set of vector spaces $TL(d) = \{TL_{2n}(d)\}$ with $n \in \mathbb{Z}_{n\geq 0}$ and modulus d. We define $TL_{2n}(d)$ to be all formal $\mathbb{C}$ linear combinations of non-crossing pairs of partitions of a rectangle, with n points on the top and n points on the bottom. For each rectangle, we shade the regions in a checkerboard pattern so that all shaded regions are separated by an unshaded region, and all unshaded regions are separated by a shaded region.

For example,



is an element of $TL_8(d)$. Moreover, we say that two Temperley-Lieb diagrams are equivalent if one can be obtained via isotopy of one or more of the strands, or possible addition or removal of a finite number of circles using the modulus relation.

The planar algebra $TL(d)$ is equipped with the following action of the planar operad: Given a planar diagram T in standard form with k input discs we have the map $\bigotimes_{i=1}^{k} TL_{2l_i}(d) \to TL_{2o}(d)$. That is, given an appropriate element of $TL_{2l_i}(d)$ for each i, our map outputs a diagram in $TL_{2o}(d)$ by inserting each picture into the appropriate rectangle, removing the rectangles, and removing any closed strings by multiplying by d. We call this type of action the "insertion action."

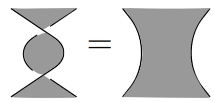
The shaded planar algebra $TL(d)$ is embedded inside every shaded planar algebra with modulus d by considering the planar diagrams with no input discs. This allows us to think of $TL(d)$ planar as being the "smallest" planar algebra with modulus d.

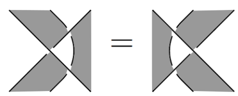
Lemma 4.2. *The dimension of $TL(d)_n$ are indexed by the Catalan numbers.*

Proof. Let $TL(d) = \{TL_{2n}(d)\}$ with $n \in \mathbb{Z}_{n\geq 0}$ and $TL_{2n}(d)$ to be all formal $\mathbb{C}$ linear combinations of non-crossing pairs of partitions of the rectangle with n points on the top and n points on the bottom. Take a rectangle and label the n points on the top from right to left with 1 to n and label the points on the bottom from $n+1$ to $2n$. Let l be an odd-numbered point and consider connecting the the lth point to an odd point

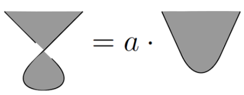
k such that $k \neq l$. This partitions the rectangle into two sets, each with an odd number of points. Since there is an odd number of points in each partition, then there is no way to make pairs of points so that each is connected by non-crossing strings. Therefore every pair of connected points must be composed of an odd point and an even point. The number of ways that these points can be connected is given by $\frac{1}{n+1}\binom{2n}{n}$, which is the equation for the nth Catalan number. Therefore, the dimensions of the box-spaces of $TL(d)$ are indexed by the Catalan numbers. □

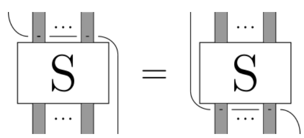
In the next few examples, we will explore shaded planar algebras generated by a braiding. The following planar algebras are all equipped with the following relations, called the Reidemeister II and III moves:

Reidemeister II (R2): 

Reidemeister III (R3): 

We define a shaded planar algebra generated by a braiding as follows

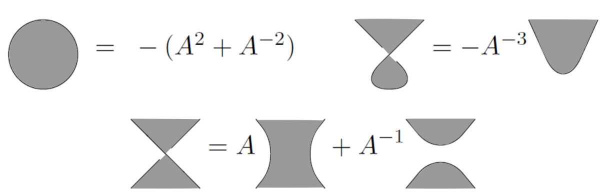
Definition 4.3. A shaded planar algebra $\mathcal{V}$ is braided if it contains a diagram satisfying R2, R3, and the twist relation:  In addition, this element must satisfy the following naturality condition:



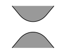
Note this definition of a braiding matches the categorical definition of a braiding. Although in this paper, we use “braiding" and “crossing" interchangeably,these terms are not always equivalent.

$TL(d)$ can also be thought of as a planar algebra generated by a braiding. This planar algebra, although isomorphic to $TL(d)$ is often called the Temperley-Lieb-Jones planar algebra because of its relationship to the Jones polynomial:

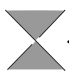
Definition 4.4. The shaded TLJ planar algebra is a shaded planar algebra generated by the braid  satisfying the following relations:

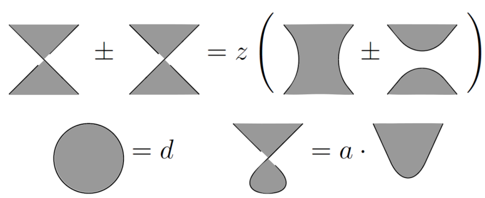


in addition to R2 and R3 and the relations with the alternate shading.

Looking at the final relation, we call  the identity, and  the cupcap. Because we consider $TL(d)$ to be the smallest planar algebra, we will always assume these diagrams are linearly independent. The TLJ planar algebra is a quotient planar algebra of tangles. The planar algebra of tangles is similar to planar algebras generated by a braid. We will not explore the planar algebra of tangles here, the reader can refer to [Edg19] for more information on this.

We will now look at the shaded Kauffman/Dubrovnik polynomial planar algebra, which is a generalization of the shaded TLJ planar algebra.

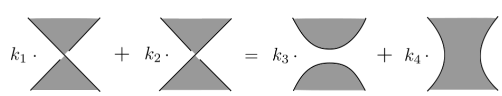
Definition 4.5. The shaded Kauffman/Dubrovnik polynomial planar algebra is a shaded planar algebra generated by the braid . The shaded Kauffman/Dubrovnik polynomial planar algebra must satisfy the following relations:



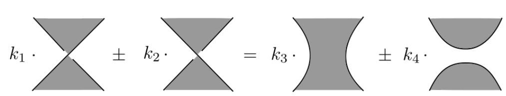
in addition to R2 and R, and these relations with the alternate shading.

Lemma 4.6. *Let $\mathcal{V}$ be a shaded planar algebra generated by the crossing with $\dim V_4 \leq 3$. Then $\mathcal{V}$ is either the shaded TLJ planar algebra or the shaded Kauffman/Dubrovnik planar algebra.*

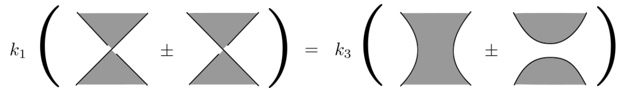
Proof. Let $\mathcal{V}$ be a shaded planar algebra generated by a crossing with $\dim V_4 \leq 3$. Since the cupcap and identity must be linearly independent, there is no planar algebra generated by a crossing where $\dim V_4 = 1$. Thus, we only need to consider $\dim V_4 = 2$, $\dim V_4 = 3$. By the definition of planar algebras generated by a crossing, $\mathcal{V}$ must satisfy the twist relation along with Reidemeister I and II and have modulus d. First let $\dim V_4 = 3$. Then there must exist a relation of the following form where k_1 and k_2 are non-zero:



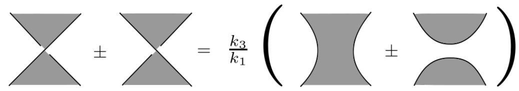
where $k_1, k_2, k_3, k_4 \in \mathbb{C}$. If we then rotate this relation by $90°$ and shade it in the alternate way we get



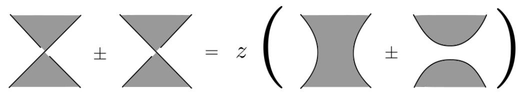
This implies that $k_1 = \pm k_2$ and $k_3 = \pm k_4$. Thus



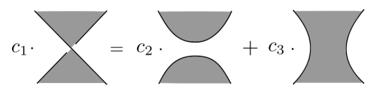
As $k_1 \neq 0$, we can divide by k_1 giving us



If we let $z = \frac{k_3}{k_1}$, then we have the relation for the shaded Kauffman/Dubrovnik planar algebra:

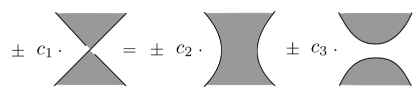


Now let $\dim V_4 = 2$. Then $\{\text{cupcap}, \text{identity}\}$ forms a basis for $\mathcal{V}$. Therefore we can express the crossing as a linear combination of the cupcap and the identity, giving us



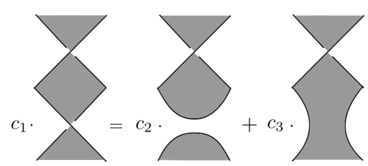
(1)

where $c_1, c_2, c_3 \in \mathbb{C}$. Rotating line (1) by 90° and re-shading we get

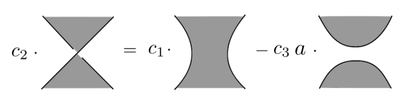


(2)

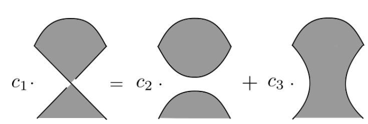
Which implies that $c_2 = \pm c_3$. Then if we vertically stack the anti-crossing on top of the relation in line (1), we have



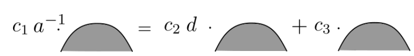
Which reduces to



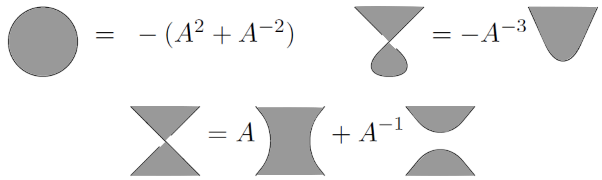
This along with the anti-crossing relation given in line 2 implies that $\frac{c_1}{c_2} = \pm\frac{c_3}{c_1}$ and $\frac{-c_3 a}{c_2} = \pm\frac{a_2}{a_1}$. If we vertically stack a cap on top of (1) we get



which reduces to



Thus $c_1 a^{-1} = c_2 d + c_3$. Solving the system of equations $c_2 = \pm c_3$ $c_1 a^{-1} = c_2 d + c_3$, $\frac{c_1}{c_2} = \pm\frac{c_3}{c_1}$, and $\frac{-c_3 a}{c_2} = \pm\frac{a_2}{a_1}$ and the normalizing, we get the following relations in terms of A:



Therefore, every shaded planar algebra generated by the crossing where $\dim V_4 \leq 3$ is the shaded Kauffman/Dubrovnik planar algebra when $\dim V_4 = 3$, or the shaded TLJ planar algebra when $\dim V_4 = 2$. □

4.1 GPA($*_n$) and spin models

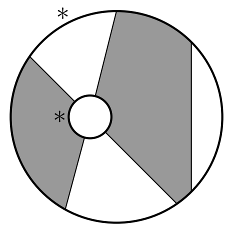
In this section, we talk about another class of shaded planar algebras, the graph planar algebra of the star graph. This is a subset of the more general notion of the graph planar algebra of a bipartite graph first defined by Jones in [Jon01].

Definition 4.7. Let S be a finite set of atoms, and let $*_n$ be the set of words of length k, made up of atoms from S. The planar algebra GPA($*_n$) is a collection of vector spaces $\{\text{GPA}(*_n)_{\pm k}\}$, where for $k \geq 1$ $\{\text{GPA}(*_n)_{\pm k}\}$ is the set of linear functionals on words from S of length k. For $k = 0$, $\text{GPA}(*_n)_{+0} = \text{GPA}(*_n)_{+1}$ and $\text{GPA}(*_n)_{-0} = \mathbb{C}$. All vector spaces are taken together with the action of the planar operad defined below.

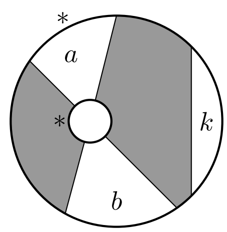
A basis for $\text{GPA}(*_n)_{\pm k}$ is the set of Kronecker deltas of words of length k. This is because, given a functional $f \in \text{GPA}(*_n)_{\pm k}$,

$$f = \sum_{l} f(l) \cdot \delta_l$$

where l ranges over all words of length k of S. We define the action of the planar operad on GPA($*_n$) on Kronecker deltas and extend linearly. Let $\{l_1, \ldots, l_n\}$ be a word of appropriate length in S. Given a planar diagram T with input discs $T_1, \ldots T_n$ we observe the action by labeling the unshaded regions of T_i (starting with the first unshaded region after the starred region) with the letters of l. Given these labelings, $T(\delta_{l_1}, \ldots, \delta_{l_n}) = \sum_l \delta_l$, where l ranges over all possible labelings of the regions bordering the output disc starting with the starred region. A labeling is possible if it is consistent with the labelings of the regions bordering the input discs in the manner described above. We exhibit this action with an example. Consider the following planar diagram:

$T :=$ 

Let $S = \{a, b, c\}$. To find $T(\delta_{ab})$, we first label the regions of T around the input disc in the manner described above. This is shown below:

$T :=$ 

Note that *any* assignment of an atom to the far right unshaded region is consistent with the labelings around the input disc. As defined above, $T(d_{ab})$ is sum of Kronecker deltas of all possible labelings of the outside region (starting with the starred region). Thus, we see that

$$T(\delta_{ab}) = \delta_{aab} + \delta_{abb} + \delta_{acb}$$

Suppose a planar diagram has a closed string. If the region inside the closed string is shaded, the closed string can be removed without penalty. If the closed string is unshaded, the string can be removed at the cost of multiplying by n, the order of S.

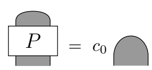
Definition 4.8. Let $\mathcal{V}$ be a planar algebra, then a spin model of $\mathcal{V}$ is defined as a map of planar algebras $\Phi : \mathcal{V} \to \mathrm{GPA}(*_n)$ for some n.

4.2 Singly-generated Yang-Baxter planar algebras

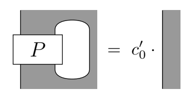
In this paper, we will be concerned with spin models of singly-generated Yang-Baxter planar algebras, which can be thought of as a slight abstraction of a planar algebra generated by a crossing. For more information about Yang-Baxter planar algebras see [Liu15].

Definition 4.9. A singly-generated Yang-Baxter planar algebra is a non-degenerate, evaluable planar algebra satisfying the following relations

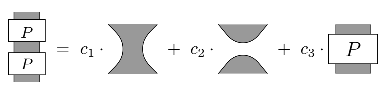
which is generated by P satisfying the following relations:



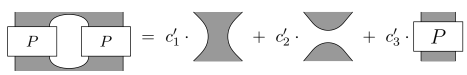
Relation 1a



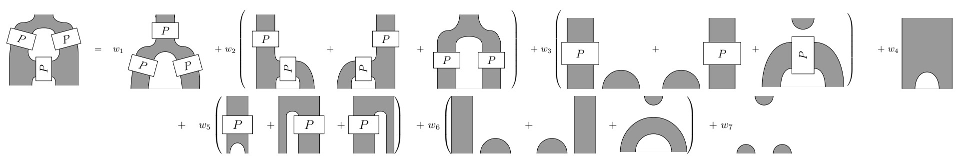
Relation 1b



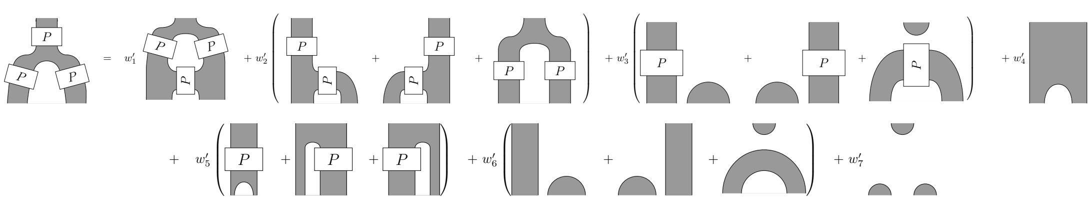
Relation 2a



Relation 2b



Relation 3a



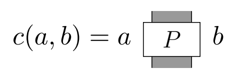
Relation 3b

To think of this as an abstraction of a planar algebra generated by a braid, we think of Relation 1a and 1b as being abstractions of the twist relation, Relation 2a and 2b as being abstractions of R2, and Relation 3a and 3b as being abstractions of R3.

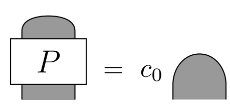
Without loss of generality, we may assume that P is a minimal idempotent, or equivalently that $P^2 = P$ and capping P on the top or bottom gives the zero diagram. By inspection, we also have another minimal idempotent, Q, with formula given by

For singly-generated YBPAs, the 3-box space contains at most two minimal idempotents (See [Edg19] for more information). Consider P and its 180-degree rotation. A priori, since the 180-degree rotation is also a minimal idempotent, it must be equal to either P or Q. For this paper, we will impose the additional relation that P is equal to its 180-degree rotation. This is equivalent to only considering symmetric spin models of singly-generated YBPAs (for more on this see [Edg19]), but this definition is omitted for brevity.

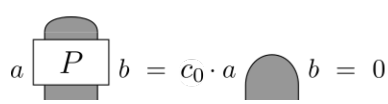
Just like with other singly-generated mathematical objects, we can define a map of planar algebras from a singly-generated YBPA, $\mathcal{V}$, by defining simply assigning P. Thus, a spin model for a singly-generated YBPA is given by $\phi(P) = \sum_{a,b \in S} c(a,b) \cdot \delta_{ab}$, where $c_{ab} \in \mathbb{C}$, which we typically think of as entries of a $n \times n$ matrix, C. Moreover, this assignment should be consistent with the relations of $\mathcal{V}$. To capture this consistency, we typically write



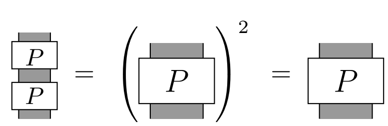
which we call the *atomized* version of P. A priori, P could be sent to any multitude of matrices C. Though, as we will see, the relations of $\mathcal{V}$ severely limit C. As an example, let us consider relations 1a and 2a. Relation 1a tells us that



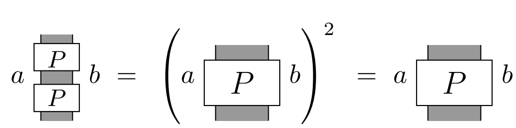
Thus, the atomized version of this relation is



Because P is a minimal idempotent, $c_0 = 0$ necessarily. Thus, this tells us that $c(a,a) = 0$ for all $a \in S$. Since P is idempotent this tells us that

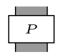
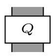


which has the corresponding atomized version:



Hence, we see that $c(a,b)^2 = c(a,b)$ for all $a, b \in S$, and $c(a,b) \in \{0,1\}$.

Because of our requirement that P is equal to its 180-degree rotation, we see that $c(a,b) = c(b,a)$ for all $a, b \in S$. Thus, relation 1a, relation 2a, and the symmetry relation tell us that the matrix C is a symmetric matrix composed of 0s and 1s, with 0s on the diagonal. In other words, C is the adjacency matrix of some graph Γ! Thus, a classification of spin models for singly-generated YBPAs is equivalent to a classification of the possible graphs represented by the adjacency matrix C.

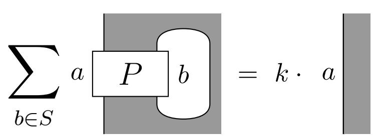
Thus, the atomized diagram a  b $= 1$ if a and b share an edge and 0 if they do not. Similarly, a  b $= 0$ if a and b share an edge and 1 if they do not. Thus, if C_P represents the adjacency matrix of Γ, C_Q represents the adjacency matrix of Γ^c. In this way, Γ and Γ^c give the same spin model for the same planar algebra up to a choice of P or Q. Therefore, we will only consider graphs up to complementation.

Definition 4.10. A graph Γ gives a spin model for a singly-generated YBPA, $\mathcal{V}$, if the adjacency matrix of Γ is C_P or C_Q.

We can think of the remaining relations of $\mathcal{V}$ as putting restrictions on Γ. As an example of this, consider what restriction relation 1b and 2b put on Γ (taken from [Edg19]).

Theorem 4.11. *Let Γ be a graph and let $\mathcal{V}$ be a singly generated YBPA. Then Γ is regular if and only if it satisfies relation 1b and strongly regular if and only if it satisfies relation 2b.*

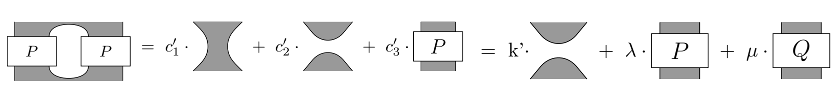
Proof. Let Γ be a graph that satisfies relation 1b. Then the atomized version of this relation is



(3)

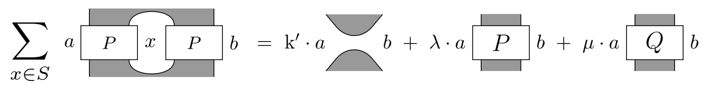
for some $k \in \mathbb{C}$. This means that given any point b adjacent to a, the value of k is independent of the choice of both a and b. Hence the right hand side of the equation tells us that the number of common neighbors between a and b is k, no matter the choice of b. This means that the number of common neighbors between any two adjacent points is constant, which implies that Γ is regular by Definition 2.8.

For the other direction let Γ be a regular graph, such that each vertex has k neighbors. Then by the Definition 2.8, for any vertices a and b, we have that equation 3 holds. This implies that Relation 1b is satisfied. Now let Γ be a graph that satisfies relation 2b. Relation 2b tells us that



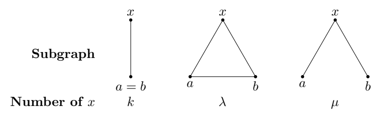
(4)

Let S be the set of vertices in Γ. Thus for any $a, b \in S$, it follows that the atomized version of relation 2b is



(5)

Notice that the right side of equation 5 counts the number of subgraphs of the following types:



From the planar diagrams in 5, it follows that when a and b are the same vertex the number of vertices x that are adjacent to a is given by the coefficient of cupcap, which is k', implying that $k' = k$. Similarly when a is adjacent to b, the number of vertices x that are adjacent to both a and b is given by the coefficient of P, which is λ. Finally when a and b are not adjacent, the number of x adjacent to both a and b is given by the coefficient of Q in the equation, which is μ. Thus by Definition 2.8, Γ is strongly regular. Now assume that Γ is strongly regular with the parameters (n, k, λ, μ). Then equation 5 holds for any pair of vertices a and b in Γ. Therefore Γ satisfies relation 2b. □

In [Edg19], the following theorem is proven:

Theorem 4.12. *Let $\mathcal{V}$ be a singly-generated Yang-Baxter planar algebra with generator P. Then $\mathcal{V}$ has a symmetric spin model if and only if the matrix of weights of P is the adjacency matrix of a graph Γ, and Γ is (up to complementation) one of the following:*

- *the pentagon,*
- *a disjoint union of complete graphs,*
- *or a 3-point regular graph with $q_3 - 3q_2 + 3q_1 - q_0 \neq 0$, where the q_i are parameters of the graph.*

Moreover we have the following corollary from that theorem (also found in [Edg19]).

Corollary 4.13. *Let Γ give a symmetric spin model for a singly-generated Yang-Baxter planar algebra, $\mathcal{V}$. Then $\mathcal{V}$ is a TLJ planar algebra when Γ or Γ^c is complete, a Bisch-Jones planar algebra when Γ or Γ^c is a disjoint union of at least two K_n, $n > 1$, and is a Kauffman polynomial planar algebra otherwise.*

A consequence of this corollary is that we can categorize the graphs that give particular 3-box space dimensions. A table of these graphs (up to complementation) and their associated dimensions are shown below:

Graph	$\dim V_3$
K_n	5
the square	10
$K_{n,n}, \quad n > 2$	11
$mK_n, \quad m, n > 2$	12
the pentagon	13

There are a number of graphs for which $\dim V_3 = 14$ or 15. These include any 3-point regular graph not listed above where $q_3 - 3q_2 + 3q_1 - q_0 \neq 0$. In the case where $\dim V_3 = 14$, these graphs are all triangle-free. Because of non-degeneracy, all graphs where $\dim V_3 = 15$ are not triangle-free. Because we are only interested in triangle-free graphs, this group of graphs will not appear in our classification later.

As we stated earlier singly generated YBPAs are an abstraction of planar algebras generated by a crossing. Thus the spin models line up with shaded planar algebras generated by a crossing. Specifically for a shaded planar algebra generated by a crossing if the $\dim V_3 = 5$ it matches up with the spin model for the shaded TLJ planar algebra. When $\dim V_3 = 13$, $\dim V_3 = 14$, or $\dim V_3 = 15$ then it correspond to the spin model for shaded Kauffman polynomial planar algebra.

In this paper, we will use the machinery developed in [Edg19] to classify certain types of 4-point regular graphs. The following theorem shows that we can do this necessarily.

Corollary 4.14. *Let Γ be a 4-point regular, triangle-free graph. Then Γ gives a spin model for a singly-generated Yang-Baxter planar algebra.*

Proof. Lemmas 2.19 through 2.23 tell us that any 4-point regular triangle-free graph, falls into one of the categories in Theorem 4.12. Thus, they all give spin models for singly-generated YBPAs, as desired. □

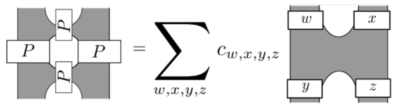
5 Classification of 4-point regular triangle-free graphs

In this section, we will prove the following theorem:

Theorem 5.1. *Let Γ be a 4-point regular triangle free graph. Then Γ is one of the following: A complete bipartite graph, a collection of bars, a collection of vertices without edges, or the pentagon.*

The following proposition follows the same style as in [Edg19]:

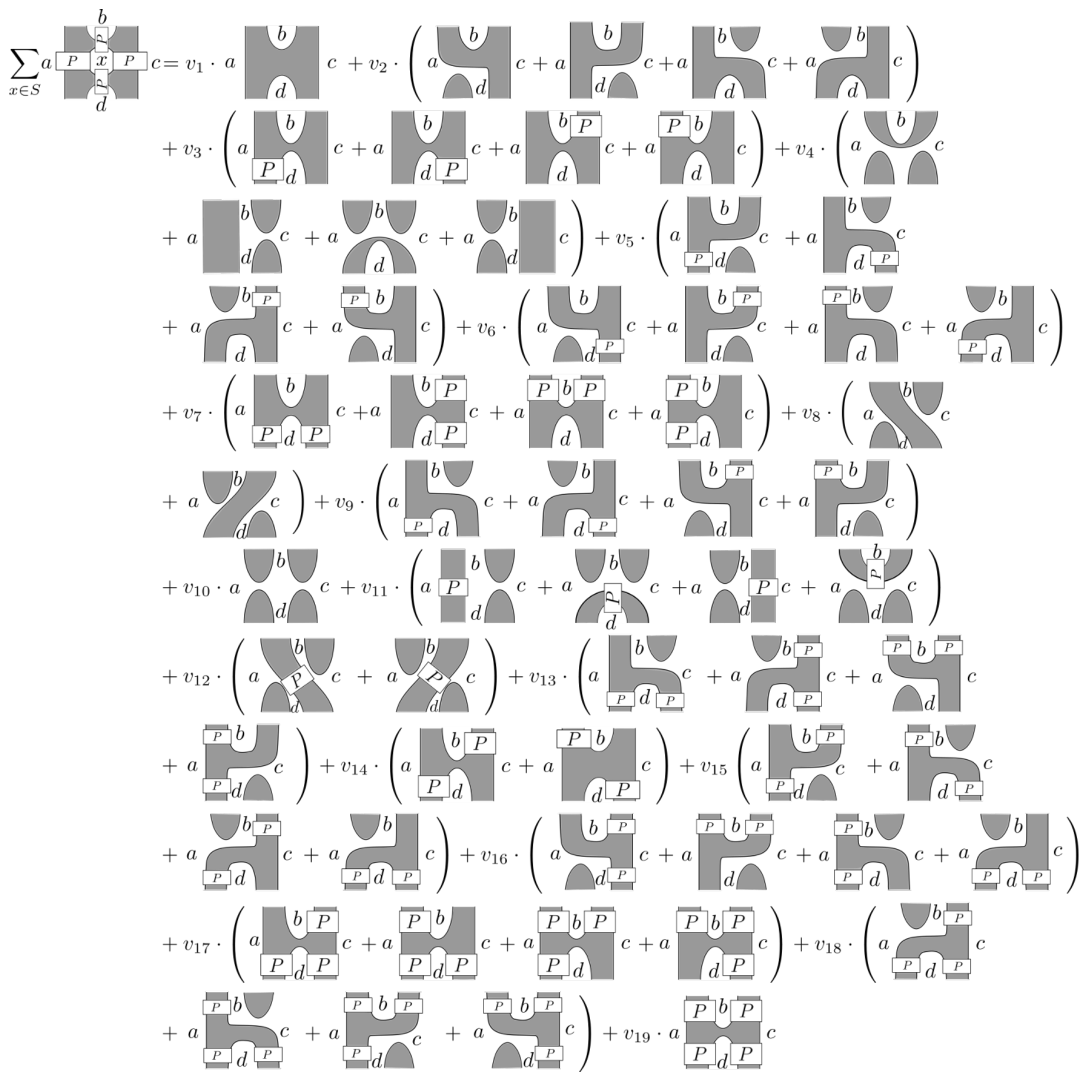
Proposition 5.2. *A graph Γ is a 4-point regular, and triangle-free if and only if Γ satisfies the following relation*



Relation 4a

where $w, x, y, z \in$ .

Proof. Let Γ be a triangle-free graph that satisfies the Relation 4a. The atomized version of this relation tells us

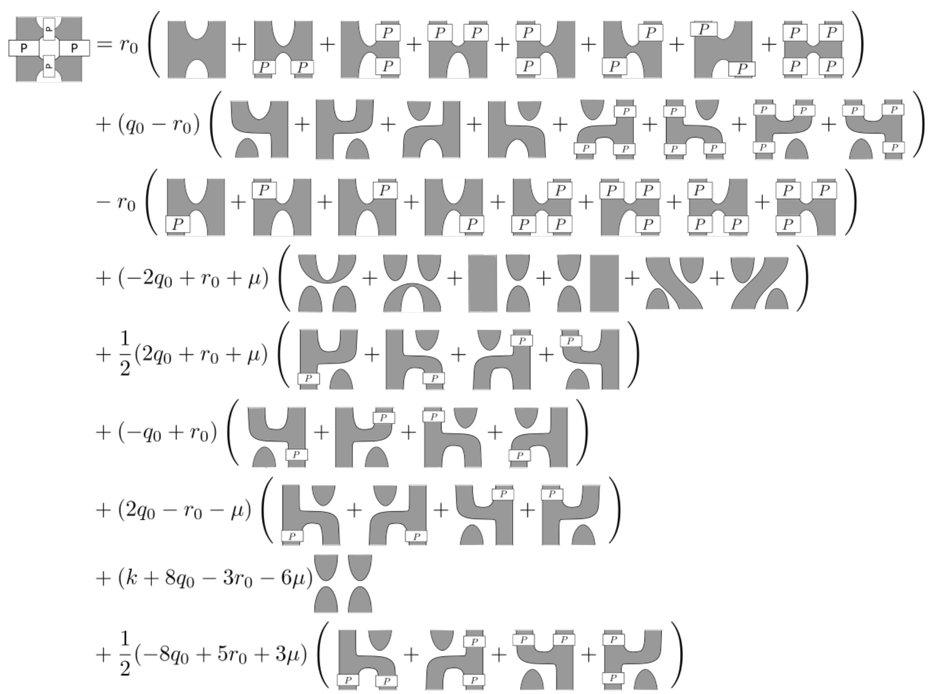


(6)

The left side of (6) counts the number of vertices in Γ that are adjacent to four vertices in Γ. The right side of (6) represents all possible ways in which the four vertices can be connected, given by the subgraphs corresponding to the parameters k, μ, q_0, r_0. Thus, the number vertices adjacent to any four vertices is determined by how the four vertices connect to one another, which implies that Γ is 4-point regular.

Now assume that Γ is 4-point regular and triangle-free graph with parameters $k, \mu, \lambda, q_0, r_0$. Therefore for any vertices a, b, c and d in Γ equation 6 holds. In conclusion Γ satisfies Relation 4a. □

In a manner similar to [Edg19], in relation 4a we can explicitly solve for the coefficients v_i in terms of the parameters of the graph. This can be found in the Appendix (See system of equations 1). Noticing that only k, μ, q_0, and r_0 are non-zero, the result can be reduced. This simplification is in the Appendix (See system of equations 2). Thus, we obtain the relation for a 4-point regular triangle free graph:



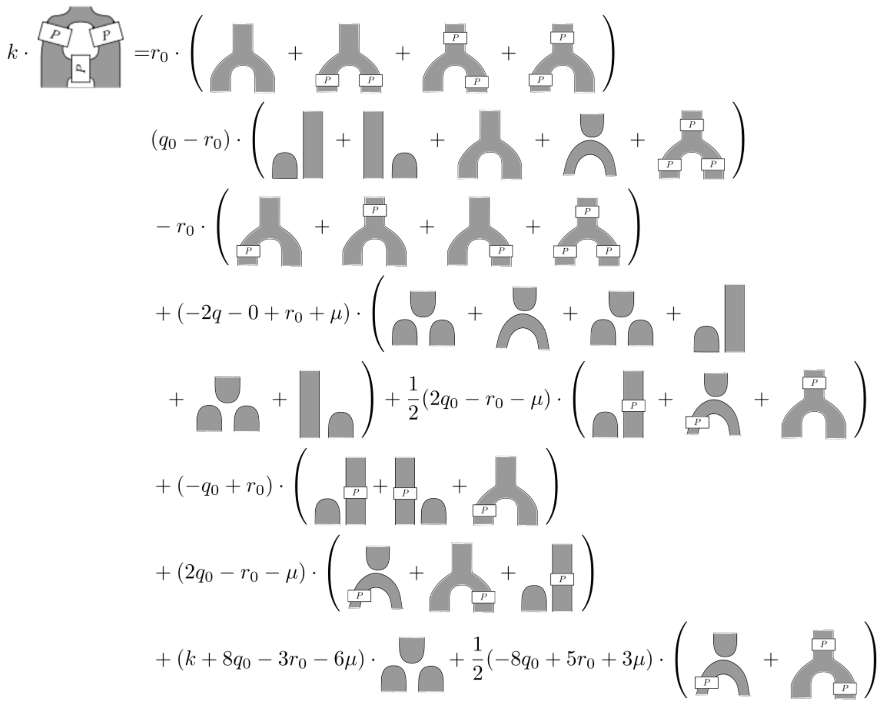
(7)

Proposition 5.3. *There does not exist a 4-point regular triangle-free graph Γ which gives a spin model of a singly generated YBPA, where the dimension of the 3-box space is 15.*

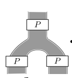
Proof. Let Γ be a 4-point regular triangle free graph which gives a spin model for a singly generated YBPA, where the dimension of the 3-box space is 15. By Corollary 4.13 there does not exist a triangle-free graph that gives a spin model where the dimension of the 3-box space is 15. Because 4-point regularity implies 3-point regularity we conclude that there does not exist a 4-point regular triangle free graph where the dimension of the 4-box space is 15. □

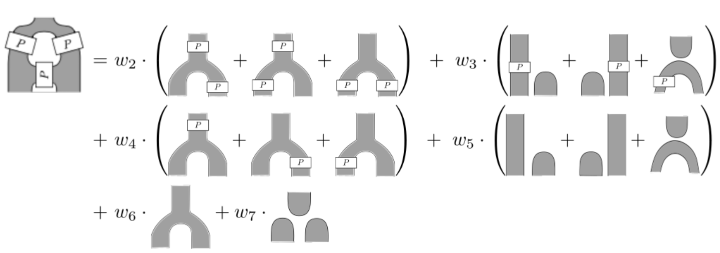
Proposition 5.4. *There does not exist a 4-point regular triangle-free graph Γ that gives a spin model for a singly generated YBPA, where the dimension of the 3-box space is 14.*

Proof. Let Γ be a 4-point regular triangle-free graph which gives a spin model for a singly-generated YBPA $\mathcal{V}$ with a 3-box space of dimension 14. Proposition 5.2 implies that (7) must hold. Notice up to rotation there are exactly two distinct ways to cap the diagram on the left. Capping every diagram in (7) on the top right corner gives us



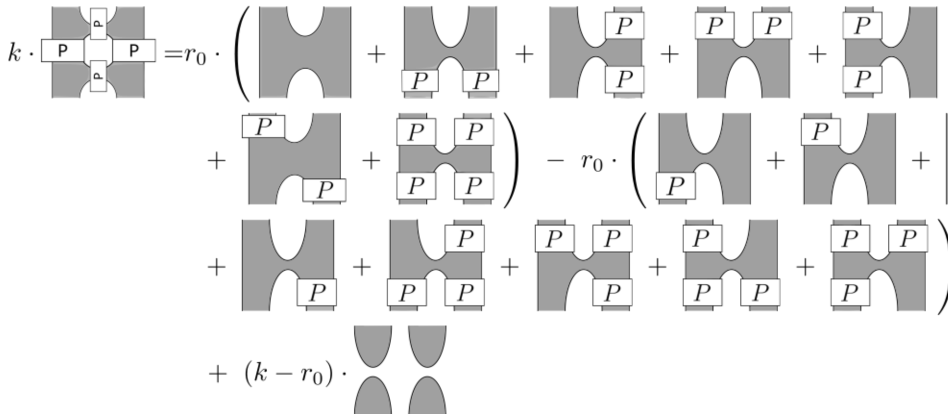
(8)

We check the linear independence of these diagrams by finding the null space of the degeneracy matrix. This matrix can be found in the Appendix under *Degeneracy matrix 1*. From this we get that the null space only contains . Thus excluding this diagram, Corollary 4.13 tells us that the remaining 14 diagrams are linearly independent. In [Edg19] it was shown that



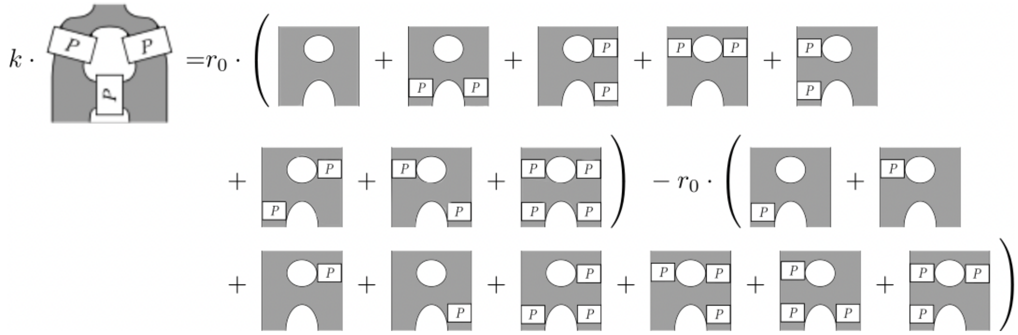
(9)

where every diagram on the right side of (9) is linearly independent. Setting (8) and (9) equal tells us that $r_0 = q_0 = \mu$ necessarily. Therefore we get the following simplified equation:



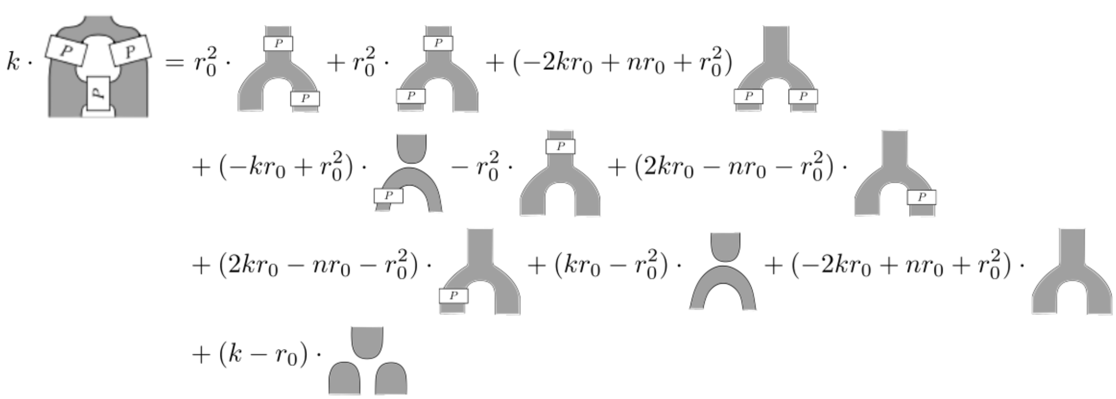
(10)

We can now cap the top center of every diagram in (10). After performing this capping we have:



(11)

Simplifying we get



(12)

Setting (12) and (9) multiplied by k equal to each other, since diagrams on both sides of the equations are linearly independent, this tells us that the diagrams that match on both sides of the equation have equal coefficients. This yields a system of equations which when solved gives us two cases: Either $r_0 \neq 0$ or $r_0 = 0$. If $r_0 \neq 0$ then $n = 2k$ and $k = r = q_0 = \mu$. By Lemma 2.11 graphs with these parameters are complete bipartite graphs, which by Corollary 4.13 give a symmetric spin model for a planar algebra where the 3-box space is dimension 11, a contradiction to $\dim V_3 = 14$. Now, if $r_0 = 0$ by Lemma 2.16, then $k < 4$ necessarily. Furthermore, Lemmas 2.20 through 2.22 tell us that Γ is either a graph with no edges, a collection of bars, the square, the pentagon, or $K_{3,3}$. However, these spin models give planar algebras with $\dim V_3 < 14$ by Corollary 4.13, a contradiction. Thus, in all cases $\dim V_3 < 14$, so no such planar algebras exist. Therefore, no such Γ exist. □

Theorem 5.5. *Let Γ be a 4-point regular triangle-free graph that gives a spin model for a singly-generated Yang-Baxter planar algebra. Then Γ is one of the following graphs (up to complementation):*

- *A complete bipartite graph*
- *A collection of bars*
- *A collection of vertices without edges*
- *The pentagon*

Proof. Let Γ be a 4-point regular triangle-free graph that gives a spin model for a singly-generated Yang-Baxter planar algebra. Then by Propositions 5.3 and 5.4, $\dim V_3 < 14$. By Corollary 4.13, the only Γ which give spin models for singly-generated YBPA with $\dim V_3 < 14$ (up to complementation) are those in the list above. By Proposition 2.18, these are all 4-point regular, triangle-free graphs. Thus, this list must be all 4-point regular triangle-free graphs which give spin models for some singly-generated YBPA, as desired. □

Corollary 5.6. *The following is a complete list of 4-point regular, triangle-free graphs:*

- *A complete bipartite graph*
- *A collection of bars*
- *A collection of vertices without edges*
- *The pentagon*

Proof. Let Γ be a 4-point regular triangle-free graph. Then by Theorem 4.12, any such graph gives a spin model for some singly generated YBPA. By Theorem 5.5, the only 4-point regular, triangle-free graphs which give a spin model for some singly-generated YBPA is in the list above. Therefore, we conclude that the list above is a complete list of 4-point regular triangle-free graphs. □

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Appendix A

System of equations 1

$$
\begin{pmatrix}
1 & 4 & 0 & 4 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 2 & 2 & 0 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 3 & 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 1 \\
1 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \\ v_{17} \\ v_{18} \\ v_{19} \end{pmatrix}
=
\begin{pmatrix} k \\ \lambda \\ \lambda \\ \mu \\ \mu \\ q_3 \\ q_2 \\ q_1 \\ q_0 \\ r_4 \\ r_3 \\ r_{2a} \\ r_{2b} \\ r_1 \\ r_0 \end{pmatrix}
$$

System of equations 2

$$
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \\ v_{16} \\ v_{17} \\ v_{18} \\ v_{19} \end{pmatrix}
=
\begin{pmatrix}
r_0 \\
q_0 - r_0 \\
r_1 - r_0 \\
\mu - 2q_0 + r_0 \\
\frac{1}{2}(\lambda - \mu + 2q_0 - 2q_1 - r_0 + 2r_1 - r_{2a}) \\
-q_0 + q_1 + r_0 - r_1 \\
r_0 - 2r_1 + r_{2b} \\
\mu - 2q_0 + r_0 \\
\lambda - \mu + 2q_0 - q_1 - q_2 - r_0 + r_1 \\
k - 6\mu + 8q_0 - 3r_0 \\
0 \\
0 \\
\frac{1}{2}(-3\lambda + 3\mu - 8q_0 + 4q_1 + 4q_2 + 5r_0 - 4r_1 + r_{2a} - 2r_{2b}) \\
r_0 - 2r_1 + r_{2a} \\
0 \\
0 \\
-r_0 + 3r_1 - r_{2a} - 2r_{2b} + r_3 \\
q_0 - q_1 - q_2 + q_3 - r_0 + r_1 + r_{2b} - r_3 \\
r_0 - 4r_1 + 2r_{2a} + 4r_{2b} - 4r_3 + r_4
\end{pmatrix}
=
\begin{pmatrix}
r_0 \\
q_0 - r_0 \\
-r_0 \\
\mu - 2q_0 + r_0 \\
\frac{1}{2}(-\mu + 2q_0 - r_0) \\
r_0 - q_0 \\
r_0 \\
\mu - 2q_0 + r_0 \\
-\mu + 2q_0 - r_0 \\
k - 6\mu + 8q_0 - 3r_0 \\
0 \\
0 \\
\frac{1}{2}(3\mu - 8q_0 + 5r_0) \\
r_0 \\
0 \\
0 \\
-r_0 \\
q_0 - r_0 \\
r_0
\end{pmatrix}
$$

Degeneracy matrix 1

$$
\begin{pmatrix}
n^2 & k^2 & k^2 & k^2 & n & n & n & 0 & kn & kn & kn & 1 & k & k & k \\
k^2 & k^2 & 0 & 0 & 0 & 0 & k & 0 & k^2 & 0 & k^2 & 0 & 0 & k & 0 \\
k^2 & 0 & 0 & k^2 & 0 & k & 0 & 0 & k^2 & k^2 & 0 & 0 & k & 0 & 0 \\
k^2 & 0 & k^2 & 0 & k & 0 & 0 & 0 & 0 & k^2 & k^2 & 0 & 0 & 0 & k \\
n & 0 & 0 & k & 1 & n & 1 & 0 & k & k & 0 & 1 & k & 0 & 0 \\
n & 0 & k & 0 & n & 1 & 1 & 0 & 0 & k & k & 1 & 0 & 0 & k \\
n & k & 0 & 0 & 1 & 1 & n & 0 & k & 0 & k & 1 & 0 & k & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
kn & k^2 & k^2 & 0 & k & 0 & k & 0 & k^2 & k^2 & kn & 0 & 0 & k & k \\
kn & 0 & k^2 & k^2 & k & k & 0 & 0 & k^2 & kn & k^2 & 0 & k & 0 & k \\
kn & k^2 & 0 & k^2 & 0 & k & k & 0 & kn & k^2 & k^2 & 0 & k & k & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
k & 0 & k & 0 & k & 0 & 0 & 0 & 0 & k & k & 0 & 0 & 0 & k \\
k & k & 0 & 0 & 0 & 0 & k & 0 & k & 0 & k & 0 & 0 & k & 0 \\
k & 0 & 0 & k & 0 & k & 0 & 0 & k & k & 0 & 0 & k & 0 & 0
\end{pmatrix}
$$